By First Fundamental Theorem of Calculus

$\frac{d}{dk}\int_k^b f(t)dt = -f(k)$

APPENDIX

DETERMINING IMPLIED RISK-NEUTRAL DISTRIBUTIONS FROM VOLATILITY SMILES

The price of a European call option on an asset with strike price K and maturity T is given by

$$c = e^{-rT}\int_{S_T=K}^{\infty}(S_T - K)\,g(S_T)\,dS_T = e^{-rT}\int_K^\infty S_T g(S_T) - K g(S_T)\,dS_T$$

where r is the interest rate (assumed constant), S_T is the asset price at time T, and g is the risk-neutral probability density function of S_T. Differentiating once with respect to K gives

$$\frac{\partial c}{\partial K} = -e^{-rT}\int_{S_T=K}^{\infty} g(S_T)\,dS_T$$

$$\frac{\partial c}{\partial K} = e^{-rT}\left[\frac{\partial}{\partial K}\int_K^\infty S_T g(S_T)\,dS_T - \frac{\partial}{\partial K} K\int_K^\infty g(S_T)\,dS_T\right]$$
$$= e^{-rT}\left[(-K g(K)) - \int_K^\infty g(S_T)\,dS_T + K g(K)\right]$$
$$= -e^{-rT}\int_K^\infty g(S_T)\,dS_T.$$

Differentiating again with respect to K gives

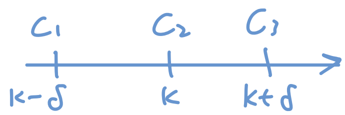
$$\frac{\partial^2 c}{\partial K^2} = e^{-rT}\,g(K)$$

This shows that the probability density function g is given by

$$g(K) = e^{rT}\frac{\partial^2 c}{\partial K^2} \tag{20A.1}$$

$$\frac{\partial^2 c}{\partial K^2} = -e^{-rT}\frac{\partial}{\partial K}\int_K^\infty g(S_T)\,dS_T.$$
$$= e^{-rT} g(K)$$

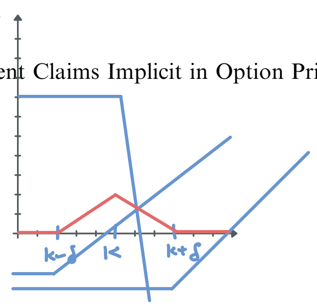
This result, which is from Breeden and Litzenberger (1978), allows risk-neutral probability distributions to be estimated from volatility smiles.[7] Suppose that c_1, c_2, and c_3 are the prices of T-year European call options with strike prices of $K - \delta$, K, and $K + \delta$, respectively. Assuming δ is small, an estimate of $g(K)$, obtained by approximating the partial derivative in equation (20A.1), is



$$e^{rT}\frac{c_1 + c_3 - 2c_2}{\delta^2}$$

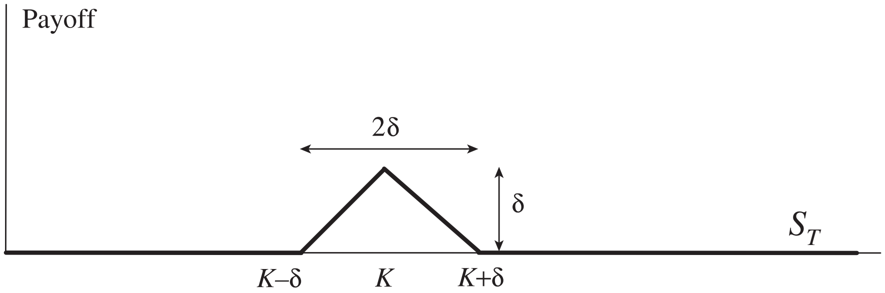
For another way of understanding this formula, suppose you set up a butterfly spread with strike prices $K - \delta$, K, and $K + \delta$, and maturity T. This means that you buy a call with strike price $K - \delta$, buy a call with strike price $K + \delta$, and sell two calls with strike price K. The value of your position is $c_1 + c_3 - 2c_2$. The value of the position can also be calculated by integrating the payoff over the risk-neutral probability distribution, $g(S_T)$, and discounting at the risk-free rate. The payoff is shown in Figure 20A.1. Since δ is small, we can assume that $g(S_T) = g(K)$ in the whole of the range $K - \delta < S_T < K + \delta$, where the payoff is nonzero. The area under the "spike" in Figure 20A.1 is $0.5 \times 2\delta \times \delta = \delta^2$. The value of the payoff (when δ is small) is therefore $e^{-rT}g(K)\delta^2$. It follows that

$$e^{-rT}g(K)\delta^2 = c_1 + c_3 - 2c_2$$



[7] See D. T. Breeden and R. H. Litzenberger, "Prices of State-Contingent Claims Implicit in Option Prices," *Journal of Business*, 51 (1978), 621–51.

Figure 20A.1 Payoff from butterfly spread.



which leads directly to

$$g(K) = e^{rT}\frac{c_1 + c_3 - 2c_2}{\delta^2} \tag{20A.2}$$

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Example 20A.1

Suppose that the price of a non-dividend-paying stock is \$10, the risk-free interest rate is 3%, and the implied volatilities of 3-month European options with strike prices of \$6, \$7, \$8, \$9, \$10, \$11, \$12, \$13, \$14 are 30%, 29%, 28%, 27%, 26%, 25%, 24%, 23%, 22%, respectively. One way of applying the above results is as follows. Assume that $g(S_T)$ is constant between $S_T = 6$ and $S_T = 7$, constant between $S_T = 7$ and $S_T = 8$, and so on. Define:

$$\begin{aligned}
g(S_T) &= g_1 \quad \text{for} \quad 6 \leqslant S_T < 7\\
g(S_T) &= g_2 \quad \text{for} \quad 7 \leqslant S_T < 8\\
g(S_T) &= g_3 \quad \text{for} \quad 8 \leqslant S_T < 9\\
g(S_T) &= g_4 \quad \text{for} \quad 9 \leqslant S_T < 10\\
g(S_T) &= g_5 \quad \text{for} \quad 10 \leqslant S_T < 11\\
g(S_T) &= g_6 \quad \text{for} \quad 11 \leqslant S_T < 12\\
g(S_T) &= g_7 \quad \text{for} \quad 12 \leqslant S_T < 13\\
g(S_T) &= g_8 \quad \text{for} \quad 13 \leqslant S_T < 14
\end{aligned}$$

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The value of g_1 can be calculated by interpolating to get the implied volatility for a 3-month option with a strike price of \$6.5 as 29.5%. This means that options with strike prices of \$6, \$6.5, and \$7 have implied volatilities of 30%, 29.5%, and 29%, respectively. From DerivaGem their prices are \$4.045, \$3.549, and \$3.055, respectively. Using equation (20A.2), with $K = 6.5$ and $\delta = 0.5$, gives

$$g_1 = \frac{e^{0.03\times0.25}(4.045 + 3.055 - 2 \times 3.549)}{0.5^2} = 0.0057$$

Similar calculations show that

$$g_2 = 0.0444, \quad g_3 = 0.1545, \quad g_4 = 0.2781$$

$$g_5 = 0.2813, \quad g_6 = 0.1659, \quad g_7 = 0.0573, \quad g_8 = 0.0113$$

Figure 20A.2 Implied probability distribution for Example 20A.1.

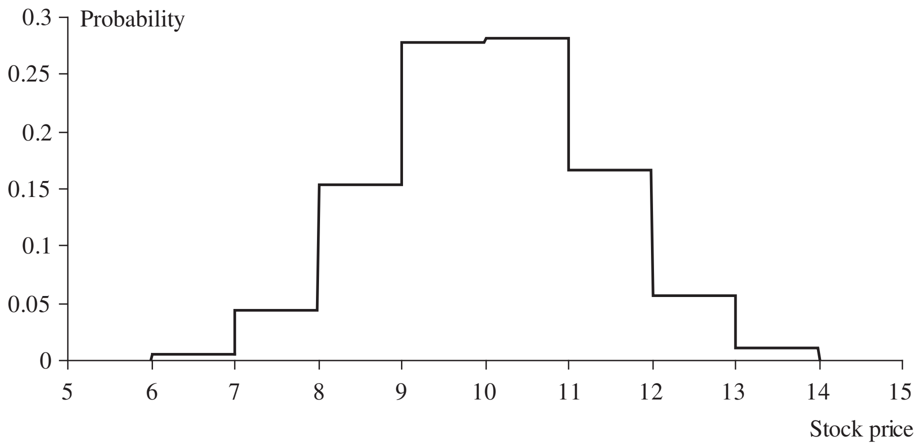

Figure 20A.2 displays the implied distribution. (Note that the area under the probability distribution is 0.9985. The probability that $S_T < 6$ or $S_T > 14$ is therefore 0.0015.) Although not obvious from Figure 20A.2, the implied distribution does have a heavier left tail and less heavy right tail than a lognormal distribution. For the lognormal distribution based on a single volatility of 26%, the probability of a stock price between \$6 and \$7 is 0.0031 (compared with 0.0057 in Figure 20A.2) and the probability of a stock price between \$13 and \$14 is 0.0167 (compared with 0.0113 in Figure 20A.2).